SGS IV
OPEN PROBLEM SESSION

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Let f be a quasiconformal reflection of the 3-sphere $\mathbb{S}^3$ and let Σ denote the fixed point set of f.

Problem 1. Is Σ quasisymmetrically equivalent to the 2-sphere $\mathbb{S}^2$?

Problem 2. Is Σ topologically equivalent to the 2-sphere $\mathbb{S}^2$?

We know from topology that Σ is a cohomological 2-sphere (Σ and $\mathbb{S}^2$ have the same cohomological groups). We can show that Σ is locally contractible (a necessary condition for Σ to be quasisymmetrically equivalent to S^2).

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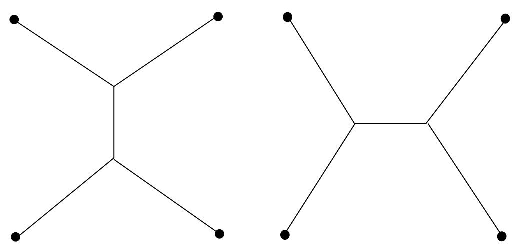
1. Length of Trees. If X is a finite tree, then we already know that there exists a configuration α_X which minimizes L over all configuration α_X on X: $L\big(\alpha_X\big) = \inf_{\alpha \in \mathcal{A}_X} L(\alpha)$.

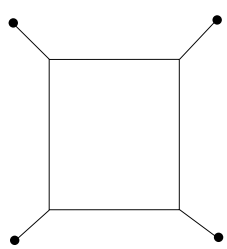
Problem: Find an algorithm to get:

$$\inf_X L(\alpha_X).$$

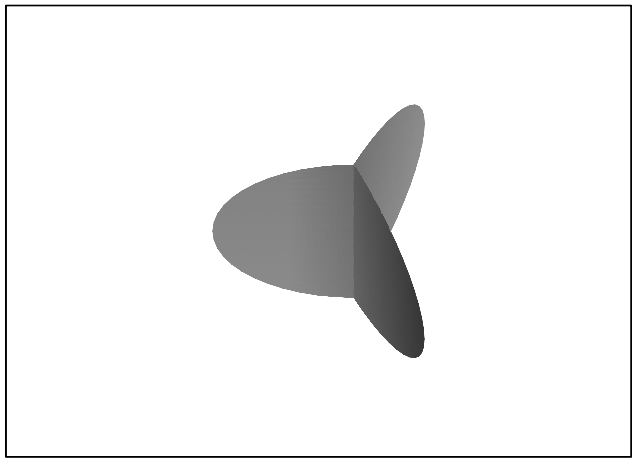
One idea, proposed by Jo Fu (University of Georgia), is to enlarge the space of admissible graphs to allow deformations from one tree to another:



One way to enable this is to allow loops, e.g.:



2. Teichmüller Space of Conformal Structures for Polyhedra. Let X^2 be a 2-dimensional topological polyhedron, such as the following:



Problem: Give a description of the Teichmüller space of X^2, similar to the description of Teichmuller spaces for plane domains or surfaces.

This description should allow one for example to formulate and prove a Gauss-Bonnet type theorem.

John McCuan
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Open Problems on Rotating Drops

Problem 1. We have shown that in the two-parameter family of rotationally symmetric rotating drops parameterized by λ and c, there is for each c a λ corresponding to a toroidal drop. Numerically, there is in fact a smooth curve $\lambda = \lambda(c)$ in the parameter space corresponding to toroidal solutions. We are unable to prove that, and it is difficult to compute derivatives of the relevant quantity (the half-period-height h) with respect to λ and c and determine its monotonicity. Furthermore, the monotonicity of h is somewhat complicated (numerically), so a delicate analysis will be required.

Problem 2. Assuming the smooth curve described above is well defined etc., we note that the two-parameter family was obtained after geometric homothety. What is the relationship between volume and λ for simply enclosed drops (no container walls) after rescaling? In particular, we know the asymptotics for all surfaces that enclose a given volume suggest non-uniqueness, i.e., there should be two geometrically different surfaces at the same rotation rate (among the spheroids and toroidal surfaces) that enclose the same volume, but we don't understand how many such surfaces there might be, etc. The same asymptotics suggest that as rotation rate increases, there is a critical rate at which there can be no equilibrium for a given volume. Is this correct? If so, what happens at this critical rate. More precisely, there should be a curve in ω (angular velocity)-λ-c space that describes equilibria for a given volume and increasing ω (starting with spheroidal solutions and then moving through toroidal solutions). What does this curve look like.

Problem 3. Prove that the toroidal solutions are unstable. (I think the stability question for the spheroids is also still open; see papers of Brulois, but the toroidal solutions may be easier.)

Problem 4. If the toroidal solutions are unstable in the classical (second variation) sense, why do we see them experimentally? Give an alternative notion of stability that allows quantitative prediction. By considering the breakup of water jets (n.b., Rayleigh), I suggest that there should be a notion of "dynamic stability" that results from the motion of the liquid.

TILLMANN GLIMM
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Problem 1. In [1] and [2], it was shown for two different reflector construction problems that in each case, the ray tracing map solving the problem is a minimizer (or a maximizer) of a certain Monge-Kantorovich cost functional. Explicitly, we have the following statement:

Let $P_0 \colon \Omega \to T$ be the ray tracing map of a reflector system solving the problem in [i] (for $i = 1, 2$). Then P_0 minimizes the transportation cost $\int_\Omega K_{[i]}(x, P(x))I(x)dx$ among all plans $P \colon \Omega \to T$.

Here we have the following domains Ω and T and cost functions:

In [1]: $\Omega, T \subseteq \mathbb{R}^n$ are bounded convex sets, $K_{[1]}(x,p) = \frac{1}{2\beta}(\beta^2 - |x-p|^2)$, $x \in \Omega, p \in T$. (Here $\beta > 0$ is some constant; note that the statement is true for any β.)
In [2]: $\Omega, T = S^n$, and $K_{[2]}(x,p) = -\log(1 - \langle x, p\rangle)$, $x, p \in S^n$.

Question: Is there a common geometrical (or physical) characterization of the two cost functions $K_{[i]}(x,p)$, $i = 1, 2$, as corresponding objects in the Euclidean (for [1]) and spherical (for [2]) geometries? If this is the case, what is the corresponding cost function in hyperbolic $n-$space, and is it possible to associate this cost function with a reflector problem as well?

Problem 2. Is it possible to apply the techniques of [1] and [2] to the problem of constructing a reflector pair that transforms a cone of rays emitted by a point source to a beam of parallel rays? (For more details see the last section of [3].) (A part of the problem is to determine whether it is possible to rewrite the inequality $\rho(x) + |\rho(x)\cdot x - (p, z(p))| + z(p) \geq \ell$ (for $x \in \Omega \subseteq S^n$, $p \in T \subseteq \mathbb{R}^n$) in the form $F(x, \rho(x)) + G(p, z(p)) \geq K(x,p)$.)

References:

[1] T. GLIMM AND V. OLIKER, *Optical design of two-reflector systems, the Monge-Kantorovich mass transfer problem and Fermat's principle*, preprint `arXiv:math.AP/0303388`, (2003).
[2] T. GLIMM AND V. OLIKER, *Optical design of one-reflector systems and the Monge-Kantorovich mass transfer problem*, J. Math. Sci. **117** (2003) no. 3, 4096–4108.
[3] V. OLIKER, *Mathematical aspects of design of beam shaping surfaces in geometrical optics*, in *Trends in Nonlinear Analysis*, Springer-Verlag (2002), 191–222.