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Flow of Hypersurfaces by Curvatures in Riemannian Manifolds

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This is a survey of results published, by many people, especially Gerhard Huisken and Ben Andrews; and unpublished, partly in collaboration with Guoyi Xu of the University of California, Irvine.

We consider the flow of an oriented hypersurface M^n in an oriented Riemannian manifold N^{n+1} by symmetric functions of the relative curvatures. For $1 \leq k \leq n$, an interesting problem is flow by Q_k-curvature, that is, $\phi : M \times [0,T) \to N$ satisfying

$$\frac{\partial \phi}{\partial t} = Q_k(\phi)\nu = \frac{S_k(k_1, \ldots, k_n)}{S_{k-1}(k_1, \ldots, k_n)}\nu,$$

where ν is the unit normal vector to $M_t = \phi(M,t)$; $k_1, \ldots, k_n$ are the principal curvatures of M_t; and S_k is the elementary symmetric polynomial of degree k in n variables. That is, S_k is the sum of $\frac{n!}{k!(n-k)!}$ terms, each term being the product of k of the n principal curvatures. For $k = 1$, $Q_1 = S_1 = k_1 + k_2 + \cdots + k_n$ is the mean curvature of M_t. For $k = n$, $Q_n = (\sum_{i=1}^n k_i^{-1})^{-1}$ is the harmonic mean curvature of M_t.

We ask for a solution which agrees with a given hypersurface M_0 at time $t = 0$.

The basic questions are

(1) Short-time existence, that is, existence for small positive T.

(2) Long-time existence: existence for a maximal time interval $0 \leq t < T_0 \leq \infty$.

(3) Behavior as $t \to T_0$: does M_t converge to a point? To a hypersurface? If the limiting set has dimension $< n$, we say that "collapsing" has occured. Must collapsing occur everywhere at the same time T_0? Is T_0 finite, or $+\infty$?

(4) Which natural quantities grow, or decrease, or remain constant, with time? Is there a formula for this growth? How is the growth affected by the curvatures of the ambient manifold N?

Certain answers will be presented for this flow in a hyperbolic $n+1$-manifold. In particular, we will show that for any integer k, collapsing may occur to a totally geodesic submanifold of any dimension $p = 0, 1, \ldots, n$. Convergence is in finite time iff $k \geq$ the codimension $n + 1 - p$. The same results are true if N^{n+1} has negatively pinched curvatures. For Q_k-flow with $1 < k < n$, most aspects of question (4) lie in the future.