Integrable equations in many-dimensional differential geometry

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Some classical problems of differential geometry can be formulated in terms of finding solutions for certain nonlinear PDE. In some cases these PDE's are integrable and can be solved by the inverse scattering method or its modifications.

The most famous problem of that sort is classification of n-orthogonal coordinate systems in flat Euclidian and pseudo-Euclidian spaces. The problem of classification of n-orthogonal coordinate systems in symmetric spaces is closely connected to the previous problem. The more general problem is classification of n-orthogonal coordinate systems in the spaces of flat connection $G(n, m)$.

All mentioned spaces belong to a general class of spaces of "diagonal curvature". By definition, the space of diagonal curvature is a Rimannian space of dimension n satisfying the following conditions:

(1) The space admits introduction of coordinates with diagonal metric tensor.
(2) In this coordinate system the Riemann curvature tensor is a diagonal matrix in the wedge tensor square of the tangent space.

The Lamé coefficients of such space satisfy the integrable system of PDE's; this system is equivalent to well-known "n-wave" system. This system has important applications to nonlinear optics.

The n-wave system is a basic example of a system integrable by the inverse scattering method. Spaces of flat connection, symmetric spaces, and flat spaces are separated from generic spaces of diagonal curvature by imposing of additional nonlinear constrains (reductions) compatible with the basic equations. The most "strong" reduction leads to vanishing of curvature and appearing of the flat space. The most "weak" reduction leads to the space of flat connection of dimension n embedded in $n + m$ dimensional flat space in the limit $m \to \infty$. This result gives a positive answer on the "Ferapontov conjecture": each space of diagonal curvature is the limiting case of spaces of flat connection. Some examples of Ricci-flat spaces of diagonal curvature (Schwarzschild and Kazner metrics) are known but the whole problem of classification of such spaces is open.