Two New Weyl-Type Bounds for the Eigenvalues of a Fixed Membrane

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary. Consider the fixed membrane problem

$$\begin{aligned} -\Delta u &= \lambda\, u \ \text{ in } \Omega, \\ u &= 0 \quad \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Denote its eigenvalues (counting multiplicity) by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \to \infty.$$

In this work, we exploit an analogy between inequalities involving λ_1 and those involving the (scaled) volume for a free membrane, (i.e., $|\Omega|^{-2/n}$) to prove two new bounds of Weyl-type for the higher pure tones of a drum:
For $k \geq 1$, we have

$$\lambda_{k+1} - \lambda_1 \leq \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n}\, \lambda_1\, k^{2/n} \tag{2}$$

and the sharper, average-type, inequality

$$\sum_{j=1}^{k} (\lambda_j - \lambda_1) \leq \frac{n}{n+2} H_n^{2/n}\, \lambda_1\, k^{1+2/n} \tag{3}$$

where

$$H_n = \frac{2\, n}{j_{n/2,1}^2 J_{n/2}^2 (j_{n/2-1,1})} \tag{4}$$

is a constant which depends on the dimension of the underlying space, n, and Bessel functions and their zeros. The results improve earlier bounds by Ashbaugh and Benguria for higher eigenvalues and highlight statements found in previous work by Laptev.