Fractional integro-differentiation by Chen-Hadamard

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The fractional differentiation by Chen-Hadamard

$$D_c^\alpha f = \frac{1}{\mu(\alpha,l)} \int\limits_0^\infty \frac{(\tilde{\Delta}_t^l f_{c+})(x) + (\tilde{\Delta}_{t^{-1}}^l f_{c-})(x)}{|ln\, t|^{1+\alpha}} \frac{dt}{t},$$

is discussed, where $\tilde{\Delta}_t^l = \sum\limits_{k=1}^{l} (-1)^k \binom{k}{l} f(x \cdot t^k)$, $l > \alpha > 0$, $c > 0, f \in L_p\left(R_+^1, \frac{dx}{x}\right)$ $\left(\text{ or } f \in L_p^{loc}\left(R_+^1, \frac{dx}{x}\right)\right)$,

$$f_{c+}(x) = \begin{cases} f(x), & x > c \\ 0, & x < c, \end{cases} \qquad f_{c-}(x) = \begin{cases} 0, & x > c \\ f(x), & x < c \end{cases}.$$

Since the Chen construction can be applied to functions with an arbitrary growth when $x \to \infty$ or $x \to 0$, this construction is more convenient when applied to such functions than the integro-differentiation by Hadamard [1] itself. As usual, the fractional derivative is to be treated as a certain limit. To this end, several types of different "truncation" of the Chen-Hadamard fractional derivative are introduced, denoted by

$$D_{c,\rho}^\alpha f, \qquad \overset{*}{D}{}_{c,\rho}^{\alpha} f, \qquad D_{c,\tilde{\rho}}^\alpha f,$$

where $\frac{1}{\sqrt[\ell]{e}} < \rho < 1$, $\tilde{\rho} = \left|ln\, \frac{x}{c}\right| ln\, \frac{1}{\rho}$, $\ell > \alpha > 0$. The following inversion theorem is valid, where $\mathcal{J}_c^\alpha \varphi$ stands for the corresponding Chen-Hadamard fractional integral.

Theorem 1. *Let* $f = \mathcal{J}_c^\alpha \varphi$, $\varphi \in L_p\left(R_+^1, \frac{dx}{x}\right)$ *(or* $\varphi \in L_p^{loc}\left(R_+^1, \frac{dx}{x}\right)$*),* $1 \le p < \infty$, $\alpha > 0$, $c > 0$. *Then*

$$\lim_{\rho \to 1} D_{c,\rho}^\alpha f = \varphi, \tag{1}$$

$$\lim_{\rho \to 1} \overset{*}{D}{}_{c,\rho}^{\alpha} f = \lim_{\rho \to 1} D_{c,\tilde{\rho}}^\alpha f = \varphi, \tag{2}$$

The limits in (1)-(2) can be understood both in $L_p\left(R_+^1, \frac{dx}{x}\right)$ or $L_p^{loc}\left(R_+^1, \frac{dx}{x}\right)$, correspondingly, $1 \le p < \infty$, except for the case $p = 1$ in (2), or almost everywhere.

Theorem 2 *For the function* $f(x)$ *to be presented as* $f(x) = (\mathcal{J}_c^\alpha \varphi)(x)$, *where* $\varphi \in L_p^{loc}(R_+^1, \frac{dx}{x}); \alpha > 0, c > 0, 1 < p < \infty$, *it is necessary and sufficient that* $|\ln \frac{x}{c}|^{-\alpha} f(x) \in L_p^{loc}(R_+^1, \frac{dx}{x})$, *and*

$$\lim_{\rho \to 1} D_{c,\tilde{\rho}}^\alpha f$$

exists in $L_p^{loc}(R_+^1, \frac{dx}{x})$.

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev. Fractional Integrals and Derivatives. Theory and Applications. *Gordon & Breach. Sci. Publ., London-N.York* (1993) (Russian Ed.:Fractional Integrals and Derivatives and Some of Their Applications. Nauka i Tekhnika, Minsk (1987).