The n-dimensional Heat Equation with Nonlinear Generalized Wentzel Boundary Conditions

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Of concern is the Cauchy problem for the heat equation on $D \subset \mathbb{R}^n$

$$u_t = \nabla \cdot (\alpha(x)\nabla u) \quad x \in D, t > 0$$
$$u(x,0) = f(x) \quad x \in \partial D$$

with nonlinear generalized Wentzel boundary conditions

$$\nabla \cdot (\alpha(x)\nabla u) + b(x)\frac{\partial u}{\partial n} \in c(x)\beta(x,u) \text{ for } x \in \partial D, t > 0.$$

The usual Wentzel boundary condition corresponds to $b(x) = c(x) = 0$. Here we assume that $\alpha \in C^1(\overline{D})$, $\alpha > 0$ on $\overline{D}$, $b, c \in C(\partial D)$, $b \geq 0$, $c \leq 0$ and β is a maximal monotone graph in $\mathbb{R}^2$ containing the origin. We present an L^p theory for this problem, which gives rise to a nonlinear semigroup on a Banach space X^p, $1 \leq p < \infty$ and on $C(\overline{D})$ in the case $p = \infty$. Hence, we obtain a solution to the initial boundary value problem. One novel feature of this work is that X^p is not $L^p(D)$ but rather an L^p space based on functions on D and on ∂D, with a measure on ∂D based on the boundary conditions. Questions of regularity and smoothing of the solutions are also obtained via Nash-Moser iteration.

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